Hindman and Milliken-Taylor Theorems for topological spaces

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A result on colorings of open covers

A *finite coloring* of A is a function $f : A \to k$ for some $k \in \omega$. $B \subset A$ is *monochromatic* if there is $i \in k$ with $f(b) = i$ for all $b \in B$.

Theorem (Tsaban 2015)

Let (X, τ) be a topological space and $f : \tau \sqcup [\tau]^2 \to k$ be a finite coloring. Suppose that X is Menger and $\mathcal{U}$ is a point-infinite open cover of X without finite subcovers. Then there are mutually disjoint finite subsets $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of $\mathcal{U}$ whose unions $V_n := \bigcup \mathcal{F}_n$ have the following properties:

- $\bigcup_{n \in H_0} V_n \neq \bigcup_{n \in H_1} V_n$ *for any finite non-empty $H_0, H_1 \subset \omega$ such that $\max H_0 < \min H_1$;*
- $f(\bigcup_{n \in H} V_n)$ *is the same for all finite non-empty $H \subset \omega$;*
- $f(\{\bigcup_{n \in H_0} V_n, \bigcup_{n \in H_1} V_n\})$ *is the same for any finite non-empty $H_0, H_1 \subset \omega$ such that $\max H_0 < \min H_1$;*
- $\{V_n : n \in \omega\}$ *covers X.*

Hindman's Theorem

For a sequence $\langle a_i : i \in \omega \rangle$ in a semigroup S, and $F \in [S]^{<\omega} \setminus \{\emptyset\}$, $F = \{i_0, \dots, i_k\} \subset \omega$ with $i_0 < \cdots < i_k$, we set
$a_F := a_{i_1} + \cdots + a_{i_k}$.
$FS\langle a_i : i \in \omega \rangle := \{a_F : F \in [\omega]^{<\omega}, F \neq \emptyset\}$.

Theorem (Hindman 1974)

For each finite coloring of ω, there exists an increasing sequence $\langle a_i : i \in \omega \rangle$ of natural numbers such that the set $FS\langle a_i : i \in \omega \rangle$ is monochromatic. $\square$

Given $p, q \in \beta S$, we let $A \in p + q$ iff
$\{b \in S : \exists C \in q \, (b + C \subset A)\} \in p$.
$p + q \in \beta S$. This extends the addition from S to βS, with the following continuity properties:

1. For every $x \in S$, the function $q \mapsto x + q$ is continuous;
2. For every $q \in \beta S$, the function $p \mapsto p + q$ is continuous.

Fact. Any compact semigroup T satisfying item 2 above has an idempotent element.

Proof. Zorn's Lemma provides us with a minimal closed subsemigroup E of T, and minimality yields $E = \{e\}$ for some idempotent $e \in T$. Indeed, fix $e \in E$. As $E + e$ is a closed subsemigroup of E, we have $E + e = E$. Thus $T := \{t \in E : t + e = e\}$ is a closed *nonempty* subsemigroup of E, and hence $T = E$. So $e + e = e$. $\square$

Proof of Hindman's Theorem by Galvin and Glazer.
Fix an idempotent element $e \in \beta\omega$. Let f be a k-coloring of ω.
Pick $i \in k$ with $A_0 := f^{-1}(i) \in e$ and set $a_{-1} = 0$.
By induction on $n \in \omega$, pick $a_n > a_{n-1}$, $a_n \in A_n$, and $A_{n+1} \subset A_n$ in e such that $a_n + A_{n+1} \subset A_n$. Considering the sums from right to left, we get that all sums $a_{i_0} + \cdots + a_{i_m}$ lie in $A_{i_0} \subset A_0$, where $i_0 < \ldots < i_m$. Thus $FS\langle a_i : i \in \omega\rangle$ is monochromatic.

The Milliken-Taylor Theorem

For $F, H \in [\omega]^{<\omega} \setminus \{\emptyset\}$, $F < H$ means $\max F < \min H$.

A *sumsequence* of $\langle a_i : i \in \omega \rangle \in S^\omega$ is a sequence $\langle a_{F_i} : i \in \omega \rangle$, where $F_i \subset \omega$ are nonempty finite and $F_i < F_{i+1}$ for all i. The relation of being a sumsequence is transitive.

$\langle b_i : i \in \omega \rangle \in S^\omega$ is *proper* if $b_F \neq b_H$ for all $F < H$ in $[\omega]^{<\omega} \setminus \emptyset$.

The *sum graph* of a proper sequence $\langle b_i : i \in \omega \rangle \in S^\omega$ is $\{\{b_F, b_H\} : F < H,\ F, H \in [\omega]^{<\omega} \setminus \emptyset\} \subset [FS\langle b_i : i \in \omega \rangle]^2$.

For a set X we consider $[X]^{<\omega}$ with the operation $\cup$ which turns it into a semigroup.

Theorem (Milliken 1975, Taylor 1976)

1. *For each finite coloring of the set $[[\omega]^{<\omega}]^2$, there are elements $F_0 < F_1 < \cdots$ in $[\omega]^{<\omega}$ such that the sum graph of $F_0, F_1, \ldots$ is monochromatic.*
2. *Let S be a semigroup, and $\langle a_i : i \in \omega \rangle \in S^\omega$. If $\langle a_i : i \in \omega \rangle$ has a proper sumsequence, then for each finite coloring of $[S]^2$, there is a proper sumsequence of $\langle a_i : i \in \omega \rangle$ whose sum graph is monochromatic.*

Item (2) is formally more general than item (1): $\{0\}, \{1\}, \{2\}$, is proper in $[\omega]^{<\omega}$.
Item (2) also follows from item (1): Wlog, $\langle a_i : i \in \omega \rangle$ is proper. Let $\phi : [S]^2 \to k$. Define $\psi[[\omega]^{<\omega}]^2 \to k$ by letting $\psi(\{F, H\}) = \phi(\{a_F, a_H\})$ for $F < H$.

Corollary

Let $\langle a_i : i \in \omega \rangle \in \omega^{\uparrow\omega}$. For each finite coloring of $[\omega]^2$, there is a proper sumsequence $\langle b_i : i \in \omega \rangle$ of $\langle a_i : i \in \omega \rangle$ whose sum graph is monochromatic. □

Hindman's Theorem follows from Milliken-Taylor.

Proposition

Let S be a semigroup, $\langle a_i : i \in \omega \rangle \in S^\omega$, and ϕ be a finite coloring of S. There is a finite coloring ψ of $[S]^2$ such that, for each proper sumsequence $\langle b_i : i \in \omega \rangle$ of $\langle a_i : i \in \omega \rangle$ with ψ-monochromatic sum graph, the set $FS\langle b_i : i \in \omega \rangle$ is ϕ-monochromatic.

Proof.
Let $\prec$ be some wellorder of $FS\langle a_i : i \in \omega \rangle$ with o.t. ω. Set $\psi(\{s, t\}) := \phi(min\{s, t\})$. Let $\langle b_i : i \in \omega \rangle$ be a proper sumsequence of $\langle a_i : i \in \omega \rangle$ with ψ-monochromatic sum graph. Given $F \in [\omega]^{<\omega}$, find $i > F$ such that $b_F \prec b_i$. Then $\phi(b_F) = \psi(\{b_F, b_i\})$ is the ψ-colour of all pairs in the sum graph of $\langle b_i : i \in \omega \rangle$ and hence does not depend on F. □

Proof of Milliken-Taylor: auxiliary staff

Let S be a semigroup. For $A \subset S$ and $\mathcal{F} \subset \mathcal{P}(S)$ let $A^*(\mathcal{F}) := \{b \in S : \exists C \in \mathcal{F}(b + C \subset A)\}$.

$\mathcal{F}$ is *idempotent*, if $A^*(\mathcal{F})$ contains an element of $\mathcal{F}$ for any $A \in \mathcal{F}$.

Example: Given $\langle a_i : i \in \omega \rangle \in S^\omega$, $\mathcal{F} = \{FS\langle a_i : i \geq n \rangle : n \in \omega\}$ is an idempotent family.

Lemma

For any idempotent $\mathcal{F} \subset \mathcal{P}(S)$ there exists an idempotent $e \in \beta S$ containing $\mathcal{F}$.

Proof.

Enought to see: $T := \{p \in \beta S : \mathcal{F} \subset p\}$ is a closed subsemigroup of βS. Closed is clear. Let $p, q \in T$ and $A \in \mathcal{F}$. Then $A^*(F)$ contains an element of $\mathcal{F} \subset p$, and hence is in p.
Since $\mathcal{F} \subset q$, $A^*(\mathcal{F}) \subset A^*(q)$, and therefore $A^*(q) \in p$. This means $A \in p + q$. $\square$

Corollary

Let $\langle a_i : i \in \omega \rangle \in S^\omega$ be a proper sequence. Then thete exists a free idempotent $e \in \beta S$ containing $\{FS\langle a_i : i \geq n \rangle : n \in \omega\}$.

Theorem (Milliken 1975, Taylor 1976)

Let S be a semigroup, and $\langle a_i : i \in \omega \rangle \in S^\omega$. If $\langle a_i : i \in \omega \rangle$ has a proper sumsequence, then for each finite coloring of $[S]^2$, there is a proper sumsequence of $\langle a_i : i \in \omega \rangle$ whose sum graph is monochromatic.

Let $e \supset \{FS\langle a_i : i \geq n\rangle : n \in \omega\}$ be idempotent. Wlog, $\langle a_i : i \in \omega\rangle$ is proper. Fix $\phi : [S]^2 \to k$. For each $s \in S$ find $i = i_s \in k$ such that
$C_i(s) := \{t \in S \setminus \{s\} : \phi(\{s,t\}) = i\} \in e$.
Define $\psi : S \to k$ by letting $\psi(s) = i_s$. Fix $M \in e$ monochromatic for ψ. Assume that the color is green. Then
$G(F) := \bigcap_{s\in F}\{t \in S \setminus \{s\} : \{s,t\} \text{ is green}\} \in e$ for each $F \in [M]^{<\omega}$.

For $D \in e$ let
$D^* := \{b \in D : \exists B \in e(b + B \subset D)\} = D^*(e) \cap D \in e$.

Observation. *Let $\langle D_n, b_n : n \in \omega\rangle$ be a sequence of pairs in $e \times S$ such that $b_n \in D_n^*$, $b_n + D_{n+1} \subset D_n$, and $D_{n+1} \subset D_n$. Then $b_{n_0} + \cdots + b_{n_m} \in D_{n_0}$ for any $n_0 < \cdots < n_m$ in ω.* $\square$

By induction on $n \in \omega$, we'll construct increasing sequences $\langle F_n : n \in \omega \rangle$ and $\langle m_n : n \in \omega \rangle$ of elements of $[\omega]^{<\omega}$ and ω, respectively, and a decreasing sequence $\langle D_n : i \in \omega \rangle$ of elements of e as follows.

Set $D_0 = M$, $m_0 = 0$, and pick *arbitrary* $\emptyset \neq F_0 \in [\omega]^{<\omega}$ such that $a_{F_0} \in D_0^*$. Possible because $FS\langle a_i : i \in \omega \rangle \in e$, so we have "$e$-many" choices.

At stage n, using $a_{F_{n-1}} \in D_{n-1}^*$, pick $B \in e$ such that $a_{F_{n-1}} + B \subset D_{n-1}$, pick $m_n > F_{n-1}$, set $D_n = D_{n-1} \cap B \cap G(FS\{a_{F_i} : i \in n\})$, and pick *arbitrary* $\emptyset \neq F_n \in [\omega \setminus m_n]^{<\omega}$ such that $a_{F_n} \in D_n^*$. Possible because $FS\langle a_i : i \geq m_n \rangle \in e$, so we have "$e$-many" choices.

By the construction, $\langle b_i = a_{F_i} : i \in \omega \rangle$ is a sumsequence of $\langle a_i : i \in \omega \rangle$.
Let $i_0 < \cdots < i_n < j_0 < \cdots j_l$, $F = \{i_0, \ldots, i_n\}$, and $H = \{j_0, \ldots, j_l\}$. Computing b_H from right to left, we see that $b_H \in D_{j_0} \subset D_{i_n+1} \subset G(FS\{b_0, \ldots, b_{i_n}\}) \subset G(b_F)$. $\square$

Milliken-Taylor for topological spaces

Intermediate Theorem. *Let* (X,τ) *be a topological space and* $f:\tau\sqcup[\tau]^2\to k$ *be a finite coloring. Suppose that and* $\mathcal{U}$ *is a point-infinite open cover of* X *without finite subcovers. Then there are mutually disjoint finite subsets* $\mathcal{F}_0,\mathcal{F}_1,\dots$ *of* $\mathcal{U}$ *whose unions* $V_n:=\bigcup\mathcal{F}_n$ *have the following properties:*

- $\bigcup_{n\in H_0}V_n\neq\bigcup_{n\in H_1}V_n$ *for any finite non-empty* $H_0,H_1\subset\omega$ *such that* $\max H_0<\min H_1$*;*
- $f(\{\bigcup_{n\in H_0}V_n,\bigcup_{n\in H_1}V_n\})$ *is the same for any finite non-empty* $H_0,H_1\subset\omega$ *such that* $\max H_0<\min H_1$*.* $\square$

Proof. Let $\mathcal{U}=\{U_n:n\in\omega\}$ be an open cover of X without finite subcovers, $S:=FS\langle U_n:n\in\omega\rangle$ with respect to $+:=\cup$. Wlog $\langle U_n:n\in\omega\rangle$ is proper: otherwise construct $\langle k_n:n\in\omega\rangle\in\omega^{\uparrow\omega}$ and $\langle x_n:n\in\omega\rangle\in X^\omega$ such that $x_n\in\bigcup_{k\in[k_n,k_{n+1})}U_k\setminus\bigcup_{k\in k_n}U_k$ and replace U_n with $\bigcup_{k\in[k_n,k_{n+1})}U_k$. $\square$

Missing from the theorem we wanted to prove:

- $f(\bigcup_{n\in H}V_n)$ is the same for all finite non-empty $H\subset\omega$;
- $\{V_n:n\in\omega\}$ covers X.

Combining the proof with a game

Given any $\mathcal{G} \subset \mathcal{P}(S)$, consider the following game $G(e, \mathcal{G})$: In the nth move, I chooses $E_n \in e$ (so $E_n \subset S$), and II responds by choosing $s_n \in E_n$. Player II wins if $\{s_n : n \in \omega\} \in \mathcal{G}$. Otherwise, player I wins.

Suppose that I *has no winning strategy* in this game. Then we can additionally get in the previous proof that $\{b_i = a_{F_i} : i \in \omega\}$, the proper sumsequence of the initial sequence of elements of S, lies in $\mathcal{G}$.

By induction on $n \in \omega$, we'll construct increasing sequences $\langle F_n : n \in \omega \rangle$ and $\langle m_n : n \in \omega \rangle$ of elements of $[\omega]^{<\omega}$ and ω, respectively, and a decreasing sequence $\langle D_n : i \in \omega \rangle$ of elements of e as follows.

Set $D_0 = M$, $m_0 = 0$, and pick *arbitrary* $\emptyset \neq F_0 \in [\omega]^{<\omega}$ such that $a_{F_0} \in D_0^*$. Possible because $FS\langle a_i : i \in \omega \rangle \in e$, so we have "$e$-many" choices.

At stage n, using $a_{F_{n-1}} \in D_{n-1}^*$, pick $B \in e$ such that $a_{F_{n-1}} + B \subset D_{n-1}$, pick $m_n > F_{n-1}$, set $D_n = D_{n-1} \cap B \cap G(FS\{a_{F_i} : i \in n\})$, and pick *arbitrary* $\emptyset \neq F_n \in [\omega \setminus m_n]^{<\omega}$ such that $a_{F_n} \in D_n^*$. Possible because $FS\langle a_i : i \geq m_n \rangle \in e$, so we have "$e$-many" choices.

By the construction, $\langle b_i = a_{F_i} : i \in \omega \rangle$ is a sumsequence of $\langle a_i : i \in \omega \rangle$.
Let $i_0 < \cdots < i_n < j_0 < \cdots j_l$, $F = \{i_0, \ldots, i_n\}$, and $H = \{j_0, \ldots, j_l\}$. Computing b_H from right to left, we see that $b_H \in D_{j_0} \subset D_{i_n+1} \subset G(FS\{b_0, \ldots, b_{i_n}\}) \subset G(b_F)$. $\square$

Milliken-Taylor for topological spaces

Let $\mathcal{U} = \{U_n : n \in \omega\}$ be an open cover of X without finite subcovers, $S := FS\langle U_n : n \in \omega\rangle$ with respect to $+ := \cup$. Wlog $\langle U_n : n \in \omega\rangle$ is proper. Assume also that X is Menger.

Goal: Construct an idempotent $e \in \beta S$ containing $\big\{FS\langle U_n : n \geq m\rangle : m \in \omega\big\}$ and consisting of open covers of X.

Then we can use the game $G(e, \mathcal{G})$ where $\mathcal{G}$ equals O, the collection of all open subcovers of $\mathcal{U}$, to ensure that the resulting sequence with monochromatic sumgraph covers X.

Consider
$\mathsf{O}_m = \{\mathcal{O} \subset S : \mathcal{O}$ contains an increasing cover $\{O_n : n \in \omega\}$ of $X\}$. O_m is a coideal. Thus $\mathsf{F} := \mathsf{O}_m^+$ is a filter.

Lemma

F *is idempotent.*

Proof. To be checked: Given any $\mathcal{F} \in \mathsf{F}$, we have
$\mathcal{F}^*(\mathsf{F}) := \big\{V \in S : \exists \mathcal{U} \in \mathsf{F}\,(\{V \cup U : U \in \mathcal{U}\} \subset \mathcal{F})\big\} \in \mathsf{F}$, i.e.,
$\mathcal{F}^*(\mathsf{F}) \cap \mathcal{O} \neq \emptyset$ for any $\mathcal{O} \in \mathsf{O}_m$.
We'll show that $\mathcal{F}^*(\mathsf{F}) \supset \mathcal{F}$. Indeed, for $V \in \mathcal{F}$ and $\mathcal{O} \in \mathsf{O}_m$ pick
$W_{\mathcal{O},V} \in \mathcal{F} \cap \big(V_0 \cup (\mathcal{O} \cap \mathcal{F})\big)$ and write it in the form
$W_{\mathcal{O},V} = V \cup U_{\mathcal{O}}$. for some $U_{\mathcal{O}} \in \mathcal{O} \cap \mathcal{F}$. Then
$\mathcal{U} := \{U_{\mathcal{O}} : \mathcal{O} \in \mathsf{O}_m\} \in \mathsf{F} = \mathsf{O}_m^+$ witnesses $V \in \mathcal{F}^*(\mathsf{F})$. $\square$

Since $\{FS\langle U_n : n \geq m\rangle : m \in \omega\}$ and $\mathsf{F} = \mathsf{O}_m^+$ are idempotent, so is their union (for a fixed A, $A^*(\mathcal{F})$ grows with $\mathcal{F}$), and hence there exists an idempotent ultrafilter $e \supset \{FS\langle U_n : n \geq m\rangle : m \in \omega\} \bigcup \mathsf{F}$.

Thus $e \supset \{FS\langle U_n : n \geq m\rangle : m \in \omega\}$ and $e = e^+ \subset \mathsf{O}_m$, and hence e consists of open covers of X. Our goal is achieved, which completes the proof of Tsaban's theorem.

The last slide

Thank you for your attention.